Lane-Emden problem: algorithm and symmetries

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Let $\Omega \subseteq \mathbb{R}^N$ be open bounded and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ for $N > 2$ and $2^* = +\infty$ if $N = 2$.

Lane-Emden problem:

$$\begin{cases} -\Delta u(x) = |u(x)|^{p-2}u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

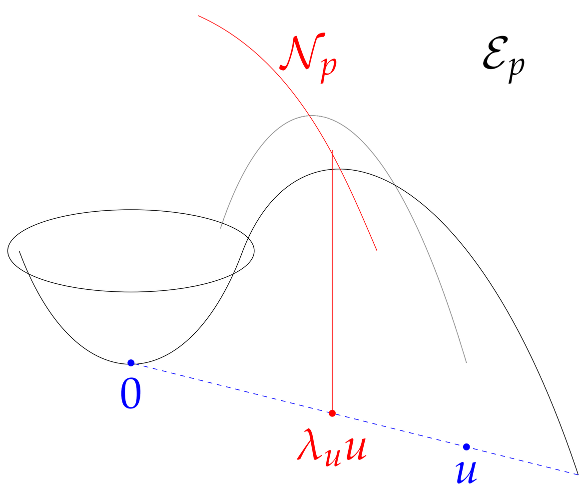
Solutions are critical points of **energy** functional

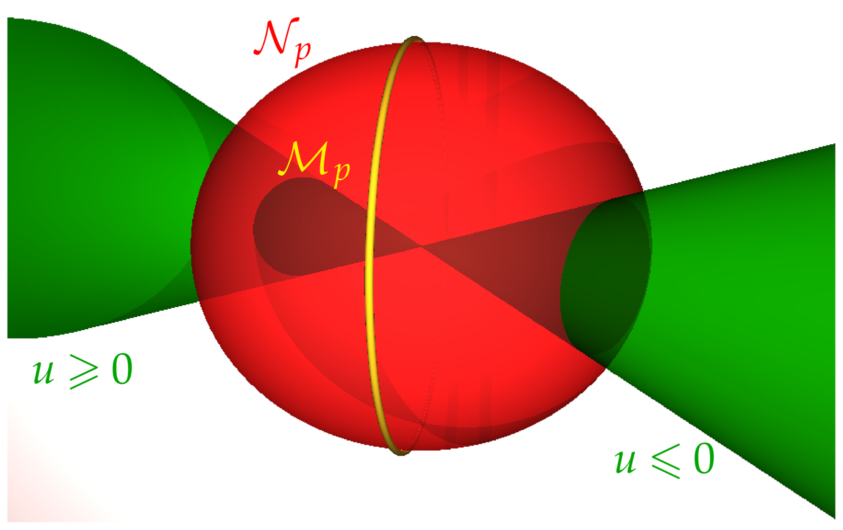
- $\mathcal{E}_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p$, where $H_0^1(\Omega) := \overline{C_0^2(\Omega)}$ for $\|u\|^2 = \int_\Omega |\nabla u|^2$.

Existence of GS and LENS:

Theorem 1: (Mountain-Pass theorem, [1] and [6])
There exist an one-signed solution and a sign-changing solution.

Sketch: Minima of $\mathcal{E}_p$ on $\mathcal{N}_p := \{u \neq 0 : \mathcal{E}_p'(u)(u) = 0\}$ and $\mathcal{M}_p := \{u : u^\pm \in \mathcal{N}_p\}$.





Mountain-Pass Algorithm: ([7] and [14])

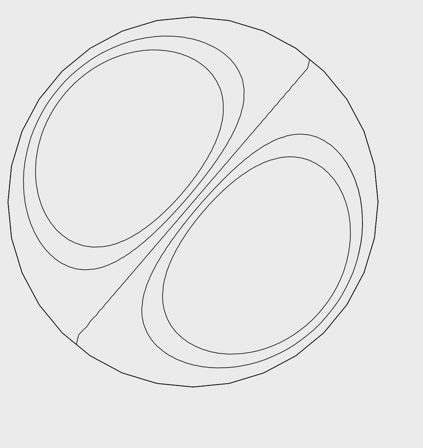
1. Let $u \in H_0^1$ and $n \leftarrow 0$;
2. Compute $u_n \leftarrow P(u)$;
3. Compute $g \leftarrow \nabla\mathcal{E}(u_n)$;
4. If $\|\nabla\mathcal{E}(u_n)\| \leqslant \varepsilon$ stop; else $v \leftarrow P(u_n - g)$;
5. If $\mathcal{E}(v) < \mathcal{E}(u_n)$, $u_{n+1} \leftarrow v$, $n \leftarrow n+1$ and go to step 3 ;
   else $g \leftarrow \frac{g}{2}$ and go to step 4;

where the projection P equals $P_N : H_0^1 \to \mathcal{N}_p : u \mapsto t^*u$ s.t. $\mathcal{E}_p(t^*u) = \max_{t>0}\mathcal{E}_p(tu)$ to approach one-signed solution and P equals $P_M : H_0^1 \to \mathcal{M}_p : u \mapsto P_N(u^+) + P_N(u^-)$ else.
Remark: A proof of convergence exists only for $P = P_N$.

Symmetries: theoretical results

- On convex domains, **GS respect** symmetries of Ω [8]. On non-convex domains, it is **not** always the case [5].
- On balls, for p small, **LENS** are even with respect to $N-1$ directions and odd with respect to the last one [3,10]. For p large, there only exists proof for even directions [2].
- On rectangles, for p small, **LENS** are even and odd with respect to a median. For p large, it is **not** always the case. We obtain a **symmetry breaking** [3].
- On squares, for p small, **LENS** are odd with respect to the center [3]. Numerically, they seem to be odd with respect to a diagonal.

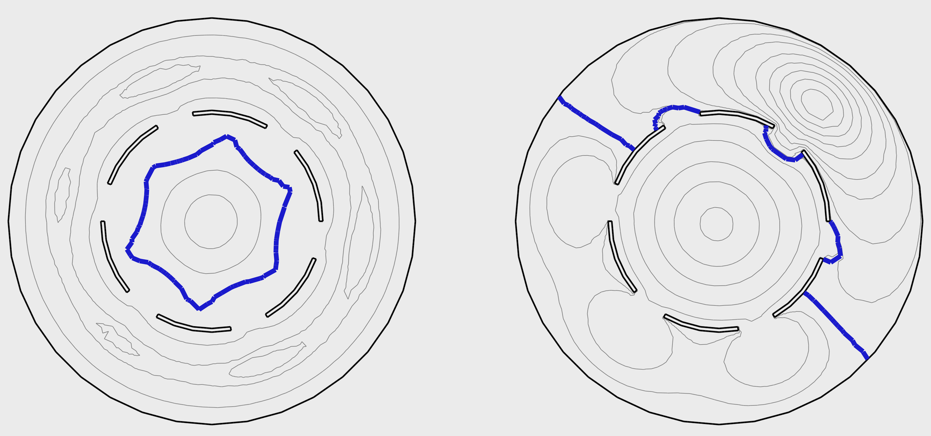
Level curves and nodal line of LENS



On a ball, for p small, the nodal line is a diameter [3,10].

When the domain is convex and p close to 2, the nodal line **touches the boundary** [12]. It is conjectured to still be true for simply connected domains.

For not connected domains, the nodal line on the domain Ω below does *not* touch $\partial\Omega$ for p small [12]. Numerically, it netherless seems touch $\partial\Omega$ for larger p.

$p = 3$  $p = 6$

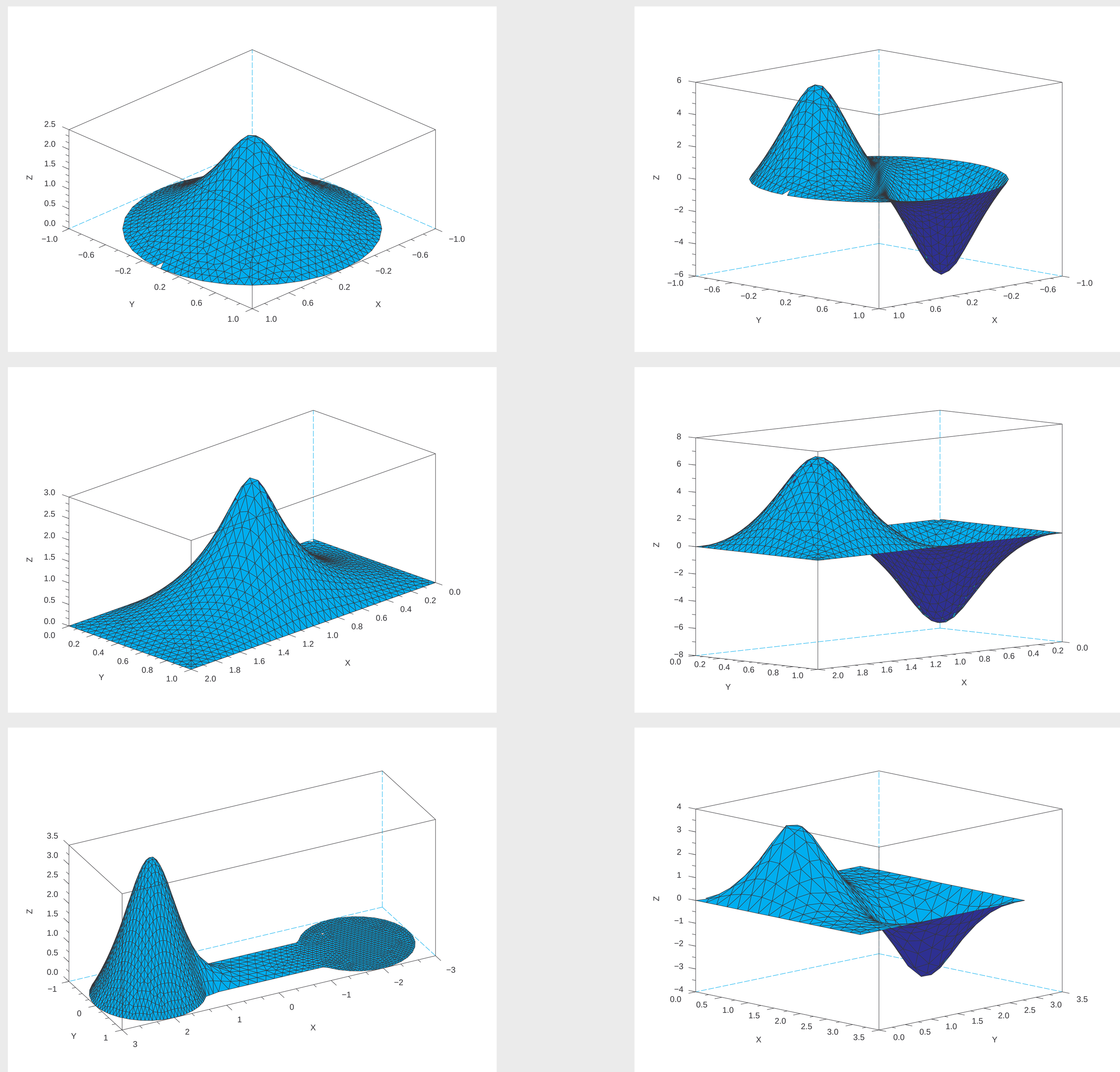
Definitions

- **Ground state** solution (**GS**): non-zero one-signed solution with minimum energy
- **Least energy nodal** solution (**LENS**): sign-changing solution with minimum energy
- $u^+ := \max(u, 0)$ and $u^- := \min(u, 0)$
- **Nodal line** of $u :=$ adh$\{x \in \Omega : u(x) = 0\}$

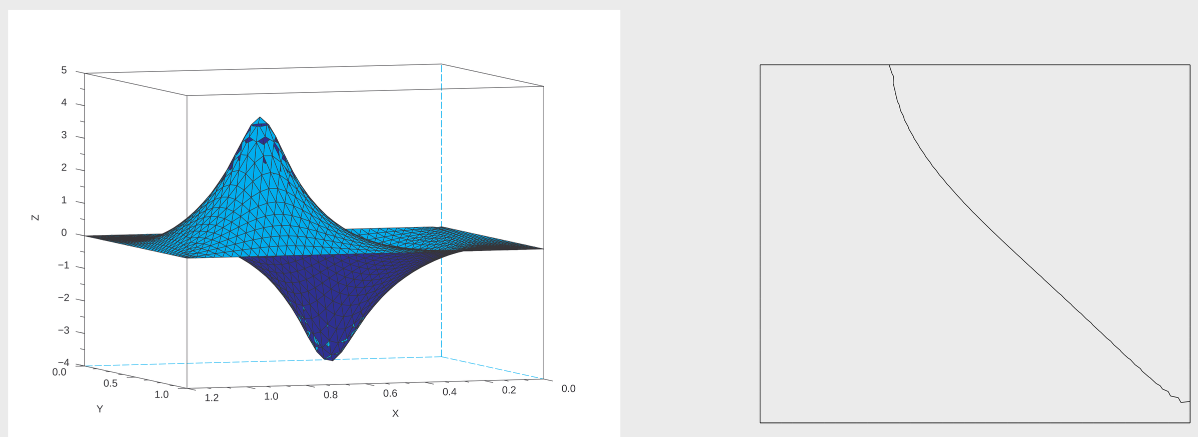
Main objectives

- Existence of **GS** and **LENS**
- **Algorithm** to approach solutions
- Theoretical study of **symmetries** of GS and LENS

Numerical solutions for $-\Delta u = u^3$



Numerical solution and nodal line for $-\Delta u = u^5$

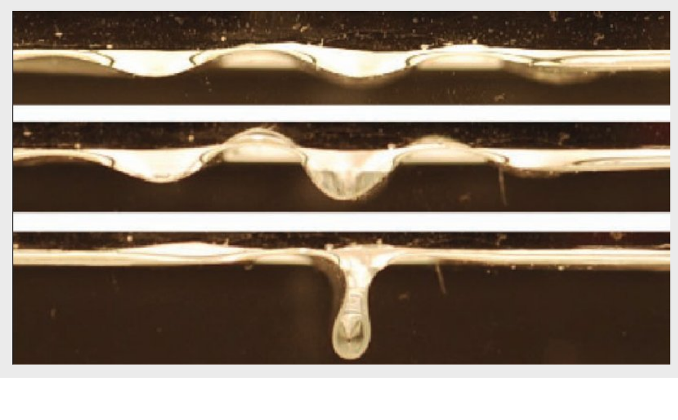


Generalizations

- Previous algorithm can be generalized to approach **GS** of problem
$$\begin{cases} -\Delta u(x) + \mathbf{V}(\mathbf{x})u(x) = |u(x)|^{p-2}u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$
when 0 does not belong to the spectrum $\sigma(-\Delta + V)$ [11,13].
Ideas: Solutions are critical points of energy functional
$$\mathcal{E}_p : H_0^1(\Omega) \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 + V(x)u^2 - \frac{1}{p}\int_\Omega |u|^p.$$
We consider $\mathcal{N}_p := \{u \neq 0 : \mathcal{E}_p'(u)(u) = 0, \mathcal{E}_p'(u)(v) = 0$, for any $v \in H^-\}$, where H^- is the negative spectrum subspace of $-\Delta + V$.
- For the q-Laplacian (i.e. $\Delta_q u = div(|\nabla u|^{q-2}\nabla u)$) problem
$$\begin{cases} -\mathbf{\Delta_q} u(x) = |u(x)|^{p-2}u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$
accumulation points, for $p \to q$, u_q of **GS** or **LENS** are functions solving $-\Delta_q u = |u|^{q-2}u$ with $u = 0$ on $\partial\Omega$ (*). If there exist non-zero solutions at (*), i.e. 1 is eigenvalue of Δ_q, u_q is different from 0 [9].
- Symmetry results work on nonlinearities $u|u|^{p-2} + (p-2)u|u|^{q-2}$, $u(e^{u^2} - 1)^{p-2}$ or $u\left(\sum_{i=1}^k \alpha_i |u|^{\beta_i(p-2)}\right)$, or for $\partial u = 0$ on $\partial\Omega$ [4].

Future works

- Symmetry breaking curves
- Symmetries, uniqueness, and multiplicity of solutions for Neumann boundary conditions and p large
- Wrinkling shapes of thin elastic films on water or polymer substrates



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