Auxiliary field method in quantum mechanics

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Introduction

Bound state problems are fundamental in quantum mechanics. Many methods are known to find analytical solutions to the Schrödinger equation: WKB, variational method, perturbation theory, …
Auxiliary field method: New efficient technique to get analytical results. Wide range of applicability.

General method

Aim: Find the spectrum of $H = T(\vec{p}^2) + V(r)$

Input: Spectrum of $H_A = T(\vec{p}^2) + g\,P(r)$
analytical ⇒ $e_A(g),\ |\psi_A(g)\rangle$
⇒ Harmonic oscillator and Coulomb

New Hamiltonian:

$$H(\rho) = T(\vec{p}^2) + \rho\,P(r) + V(I(\rho)) - \rho\,P(I(\rho))$$

with $I(x) = K^{-1}(x),\ K(x) = V'(x)/P'(x)$

The operator ρ is the auxiliary field.

Elimination: $\delta H(\rho)|_{\rho=\hat{\rho}} = 0$
⇒ $\hat{\rho} = K(r)$ and $H(\hat{\rho}) = H$ — Equivalence

Approximation: ρ is a variational parameter.

Analytical spectrum

$$E(\rho) = e_A(\rho) + V(I(\rho)) - \rho\,P(I(\rho))$$

Physical value: $E(\rho_0)$ with $\partial_\rho E(\rho)|_{\rho=\rho_0} = 0$

Properties: $\rho_0 \approx \langle\psi_A(\rho_0)|\hat{\rho}|\psi_A(\rho_0)\rangle$

Variational, depending on the form of $V(r),\ P(r)$

Schrödinger Hamiltonians

Kinetic term: $T(\vec{p}^2) = \frac{\vec{p}^{\,2}}{2m}$, many applications

Power-law potentials: $V(r) = a\,\mathrm{sgn}(\lambda)\,r^\lambda$

$$E(\rho_0) = \frac{2+\lambda}{2\lambda}\frac{(a\,|\lambda|)^{\frac{2}{\lambda+2}}}{m^{\frac{\lambda}{\lambda+2}}}N^{\frac{2\lambda}{\lambda+2}}, \qquad N = b(\lambda)n + l + c(\lambda)$$

Exact: $\lambda=2$, $N = 2n + l + 3/2$; $\lambda=-1$, $N = n + l + 1$

Exponential potential: $V(r) = -\alpha\,\mathrm{e}^{-\beta r}$

Analytical energy spectrum and finite number of bound states

Improvement: Comparison with numerical data

Relativistic Hamiltonians

Kinetic term: $\sqrt{\vec{p}^2 + m^2}$ ⇒ relativistic effects

Replacement by $\frac{\mu}{2} + \frac{\vec{p}^{\,2}+m^2}{2\mu}$

Equivalence if the auxiliary field μ is an operator.

Approximation: μ is a variational parameter.

Schrödinger-like formalism

Coulomb problem: $V(r) = -\frac{a}{r}$

$$M = m\sqrt{1 - \frac{a^2}{N^2}}, \quad N = b(a) + l + c(a)$$

Remark: Many applications in hadronic physics

$\mathcal{N}$-body systems

Interest: Crucial in molecular and atomic physics

Input: Analytical spectrum of

$$H_{\text{ho}} = \sum_i \frac{\vec{p}_i^{\,2}}{2m} + k\sum_i(\vec{r}_i - \vec{R})^2 + \bar{k}\sum_{i<j}(\vec{r}_i - \vec{r}_j)^2$$

⇒ $E_{\text{ho}} = \sqrt{\frac{2}{m}(k + \mathcal{N}\bar{k})}\ N,\ N = \sum_{i=1}^{\mathcal{N}-1}(2n_i + l_i + 3/2)$

Example: Atomic systems

$$H_{\text{at}} = \sum_i\left[\sqrt{\vec{p}_i^{\,2} + m^2} - \alpha|\vec{r}_i - \vec{R}|^{-1}\right] + \bar{\alpha}\sum_{i<j}|\vec{r}_i - \vec{r}_j|^{-1}$$

$$M_{\text{at}} = \mathcal{N}m\sqrt{1 - \frac{1}{N^2}\left[\alpha\mathcal{N} - \frac{\bar{\alpha}}{\mathcal{N}}\left(\frac{\mathcal{N}(\mathcal{N}-1)}{2}\right)^{3/2}\right]^2}$$

Other cases: Boson stars, gaussian potential, …

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